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A high-resolution scheme for the equations governing 2D bed-load sediment transport

Justin Hudson[1,*,†] and Peter K. Sweby[2,‡]

[1]*School of Mathematics, University of Manchester, Sackville Street, Manchester, M60 1QD, U.K.*
[2]*Department of Mathematics, University of Reading, Whiteknights, P.O. Box 220, Reading, Berkshire, RG6 6AX, U.K.*

SUMMARY

This paper investigates how to accurately numerically approximate the equations governing 2D sediment transport by considering two approaches: a steady and unsteady approach. A high-resolution scheme based on Roe's scheme is used to approximate both approaches with the results compared for a 2D test case. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: sediment transport; high resolution; shallow water equations; morphodynamic modelling

1. INTRODUCTION

Sediment transport plays an important role in many coastal engineering applications. The ability to obtain an accurate prediction of sediment transport is crucial to the understanding of bed morphological change and coastal water quality. Thus, it is of great importance to obtain an accurate numerical approximation of the governing equations.

In this paper, we adapt the one-dimensional sand transport model discussed by Hudson and Sweby [1] to two dimensions. The 2D model, which makes the assumption of constant porosity consists of the equation for conservation of mass,

$$h_t + (uh)_x + (vh)_y = 0 \tag{1}$$

*Correspondence to: Justin Hudson, School of Mathematics, University of Manchester, Sackville Street, Manchester, M60 1QD, U.K.
†E-mail: justin.hudson@manchester.ac.uk
‡E-mail: p.k.sweby@reading.ac.uk

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the equation for conservation of momentum in the x direction,

$$(uh)_t + \left[hu^2 + \frac{1}{2}gh^2\right]_x + (huv)_y = -ghB_x \tag{2}$$

the equation for conservation of momentum in the y direction,

$$(uh)_t + (huv)_x + \left[hv^2 + \frac{1}{2}gh^2\right]_y = -ghB_y \tag{3}$$

and the bed-updating equation,

$$B_t + \xi(q_1)_x + \xi(q_2)_y = 0 \quad \text{where} \quad \xi = \frac{1}{1-\varepsilon} \tag{4}$$

Here, $h(x,y,t)$, $u(x,y,t)$, $v(x,y,t)$, $B(x,y,t)$, $q_1(u,v,h)$ and $q_2(u,v,h)$ denote the water height, velocity in x and y direction, bed height and sediment transport rate in the x and y direction, respectively. The porosity is assumed to be constant, i.e. $\varepsilon = 0.4$. In general, the sediment transport fluxes, $q_k(u,v,h)$, are not direct functions of B, which can cause difficulties for some numerical schemes (especially if the sediment transport flux is algebraically complex).

To illustrate the numerical techniques discussed in this paper, we use the basic sediment transport flux of Grass [2], to obtain

$$q_1(u,v) = Au(u^2+v^2)^{1/2(m-1)} \quad \text{and} \quad q_2(u,v) = Av(u^2+v^2)^{1/2(m-1)} \tag{5}$$

where A is a dimensional constant that is usually determined from experimental data and m is chosen such that $1 \leqslant m \leqslant 4$, see References [3,4] for more information. In this paper, we consider fine sand ($d_{50} \approx 0.25$ mm), which results in $A = 0.001$, and use $m = 3$.

Like many inhomogeneous systems of conservation laws,

$$\mathbf{w}_t + \mathbf{F}(\mathbf{w})_x + \mathbf{G}(\mathbf{w})_y = \mathbf{R} \tag{6}$$

where $\mathbf{F}(\mathbf{w})$ and $\mathbf{G}(\mathbf{w})$ denote the flux-functions and $\mathbf{R}$ contains the inhomogeneous terms, determination of a general solution to the equations is not viable, and so numerical methods are implemented to solve them. In the next section, we extend two formulations (a steady and unsteady approach) discussed in Reference [1] to two dimensions, which are then discretised in Section 3 for a high-resolution scheme discussed by Hubbard and Garcia-Navarro [5] with the results of the two approaches compared in Section 4 for a 2D test problem.

2. STEADY AND UNSTEADY APPROACH

In this paper, we consider two approaches discussed by Cunge *et al.* [6] that can be used to approximate system (1)–(4). Both approaches are derived for the sediment transport flux (5), but they can be adapted for any sediment transport formulae required with varying degrees of difficulty. The steady approach has been researched thoroughly and is currently used in industry whereas little progress has been made on the unsteady approach due to the complexity of accurate approximation.

2.1. *Steady approach (formulation A-CV)*

The steady approach assumes that changes in the water flow are negligible, i.e. only due to the morphological changes, and the bed evolves at a much slower rate than the water flow. These assumptions allow the steady approach to take advantage of the slow evolution of the bed by decoupling the water flow from the bed update and sequentially updating the bed followed by an iteration of the 2D shallow water equations,

$$\begin{bmatrix} h \\ uh \\ vh \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}_x + \begin{bmatrix} vh \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}_y = \begin{bmatrix} 0 \\ -ghB_x \\ -ghB_y \end{bmatrix} \tag{7}$$

to an equilibrium state whilst keeping the bed fixed. For most practical applications, iterating the water flow to an equilibrium state for each bed update (steady approach) reduces the computational time taken to evolve the water flow with the bed (unsteady approach). Thus, this approach will be computationally faster for test cases where the bed evolves slowly. However, Hudson and Sweby [1] illustrated that the steady approach was only valid in 1D for a limited set of cases but for those cases it produced less diffusive results than the unsteady approach.

2.2. *Unsteady approach (formulation C)*

The unsteady approach does not make any assumptions and preserves the connection between the water flow and the sediment by discretising the equations simultaneously. In Reference [1], a rigorous comparison was made in 1D for four unsteady approaches and it was illustrated that approximating system (1)–(4) as written produced accurate results for a limited set of test cases. This may be due to the singular Jacobian matrix and is rectified by using the product rule,

$$g(Bh)_x = gBh_x + ghB_x$$

to re-write the inhomogeneous terms present in the equations for conservation of momentum (2) and (3) thus, obtaining the system

$$\begin{bmatrix} h \\ uh \\ vh \\ B \end{bmatrix}_t + \begin{bmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 + ghB \\ huv \\ \xi q_1 \end{bmatrix}_x + \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 + ghB \\ \xi q_2 \end{bmatrix}_y = \begin{bmatrix} 0 \\ gBh_x \\ gBh_y \\ 0 \end{bmatrix} \tag{8}$$

The Jacobian matrix of this system is no longer singular and tests in 1D illustrated that this system is considerably more robust than the original system (formulation A-SF). Hudson and Sweby [1] also illustrated that the two systems were conservatively equivalent and Hudson [4] illustrated that as the mesh is refined, the results of the two systems converge.

3. HIGH-RESOLUTION SCHEME

Both approaches are approximated by using a high-resolution scheme (see Reference [7] for more information) discussed by Hubbard and Garcia-Navarro [5] that is based on Roe's scheme [8]. The scheme balances the approximation of the inhomogeneous terms with the numerical flux-functions for equilibrium problems,

$$\mathbf{F}_x + \mathbf{G}_y = \mathbf{R}$$

and thus, satisfies the C-property of Bermúdez and Vázquez [9]. To ensure this balance occurs, the numerical scheme separates the inhomogeneous terms,

$$\mathbf{R} = \mathbf{f} + \mathbf{g} \quad \Rightarrow \quad \mathbf{F}_x = \mathbf{f} \quad \text{and} \quad \mathbf{G}_y = \mathbf{g}$$

where $\mathbf{f}$ and $\mathbf{g}$ contain the terms with derivatives in the x and y direction, respectively. The high-resolution scheme can now be written as

$$\begin{aligned}\mathbf{w}_{i,j}^{n+1} = \mathbf{w}_{i,j}^{n} &- s_x\left(\mathbf{F}^*_{i+1/2,j} - \mathbf{F}^*_{i-1/2,j}\right) + s_x\left(\mathbf{f}^-_{i+1/2,j} + \mathbf{f}^+_{i-1/2,j}\right) \\ &- s_y\left(\mathbf{G}^*_{i,j+1/2} - \mathbf{G}^*_{i,j-1/2}\right) + s_y\left(\mathbf{g}^-_{i,j+1/2} + \mathbf{g}^+_{i,j-1/2}\right)\end{aligned} \tag{9}$$

where

$$\mathbf{F}^*_{i+1/2,j} = \frac{1}{2}(\mathbf{F}^n_{i+1,j} + \mathbf{F}^n_{i,j}) - \frac{1}{2}\sum_{k=1}^{p}\left[\tilde{\alpha}_k^F|\tilde{\lambda}_k^F|(1 - \Phi(\tilde{\theta}_k^F)(1 - |\tilde{v}_k^F|))\tilde{\mathbf{e}}_k^F\right]^n_{i+1/2,j}$$

$$\mathbf{f}^{\pm}_{i+1/2,j} = \frac{1}{2}\sum_{k=1}^{p}\left[\tilde{\beta}_k^F\tilde{\mathbf{e}}_k^F(1 \pm \text{sgn}(\tilde{\lambda}_k^F)(1 - \Phi(\tilde{\theta}_k^F)(1 - |\tilde{v}_k^F|)))\right]^n_{i+1/2,j}$$

$$\tilde{v}_k^F = s_x\tilde{\lambda}_k^F, \quad \tilde{\theta}_k^F = \frac{(\tilde{\alpha}_k)^F_{I+1/2,j}}{(\tilde{\alpha}_k)^F_{i+1/2,j}}, \quad I = i - \text{sgn}(\tilde{v}_k^F)_{i+1/2,j}, \quad s_x = \frac{\Delta t}{\Delta x}, \quad s_y = \frac{\Delta t}{\Delta y}$$

and $\Phi(\tilde{\theta}_k)$ is the minmod flux-limiter (see Reference [10] for more information).

The scheme uses Roe averaged values, which are denoted with a $\sim$ and are calculated from the Roe decomposition (see References [4, 5, 8] for more details), where p is the number of components in the system ($p = 3$ or 4 depending on the approach). The eigenvalues, $\tilde{\lambda}$, and eigenvectors, $\tilde{\mathbf{e}}$, of the Roe averaged Jacobian matrix, $\tilde{\mathbf{A}}$, are used to obtain the wave strengths, $\tilde{\alpha}$, and the approximation of the inhomogeneous terms, $\tilde{\beta}$, from the Roe decomposition. The approximations of $\mathbf{G}^n_{i,j+1/2}$ and $\mathbf{g}^{\pm}_{i,j+1/2}$ follow in an obvious manner. To ensure the scheme remains stable, we use a variable time step

$$\Delta t = \frac{v\min(\Delta x, \Delta y)}{2\max_{i,j}(|\lambda^F|, |\lambda^G|)}$$

We now adapt the high-resolution scheme to approximate the steady and unsteady approach with the sediment transport flux (2) with $m = 3$.

For the steady approach, the system is decoupled into a water flow (5) approximation, using the Roe averages obtained by Glaister [11], which is iterated to an equilibrium state, followed by a bed update as a separate scalar equation using the approach of De Vries [12],

$$\lambda^F = \xi\left[\frac{\partial q_1}{\partial B}\right] \approx \frac{guq_u}{c^2 - u^2} \quad \text{where} \quad q_u = \frac{\partial q_1}{\partial u} = A(3u^2 + v^2)$$

(and similarly for λ^G) to calculate the wavespeed. The scheme can be adapted to approximate the unsteady approach by using the values in Reference [4] which were derived for the sediment transport flux (5).

4. NUMERICAL RESULTS

So that we can compare the two approaches in 2D, we use the following 2D test problem, which consists of a 1500 m × 1000 m channel with initial conditions,

$$h(x,y,0) = 10 - B(x,y,0), \quad u(x,y,0) = \frac{10}{h(x,y,0)}, \quad v(x,y,0) = 0$$

and the initial bathymetry is a dome of sand,

$$B(x,y,0) = \begin{cases} \sin^2\left(\frac{\pi(x-500)}{200}\right)\sin^2\left(\frac{\pi(y-400)}{200}\right) & \text{if } 500 \leqslant x \leqslant 700,\ 400 \leqslant y \leqslant 600 \\ 0 & \text{otherwise} \end{cases}$$

We compare the two approaches for the high-resolution scheme (with a CFL = 0.8 and $\Delta x = \Delta y = 20$) by running the channel test problem until $t = 200$ h. By this time, DeVriend [13] deduced that the dome would spread out into a star-shaped pattern. For this test case, the Froude number is approximately 0.1. Figures 1 and 2 show the results obtained at $t = 200$ h for the steady and unsteady approach, respectively. Both approaches have produced smooth results with no oscillations present, but the steady approach has produced kinks at the front of (and also behind) the dome. Notice that the steady approach seems to have moved the same at a slightly faster speed than the unsteady approach.

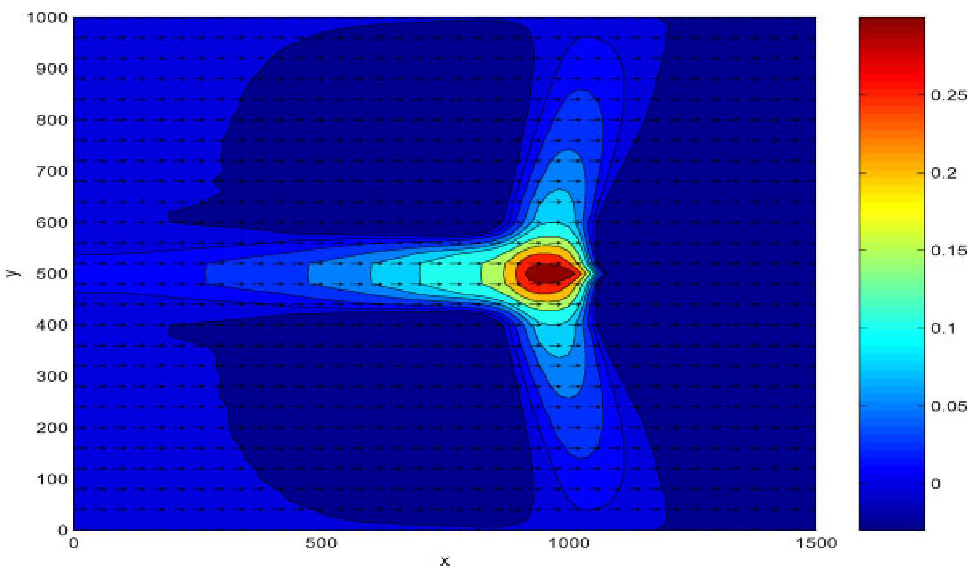

Figure 1. Contour and vector plot of the results using the steady approach at $t = 200$ h (B).

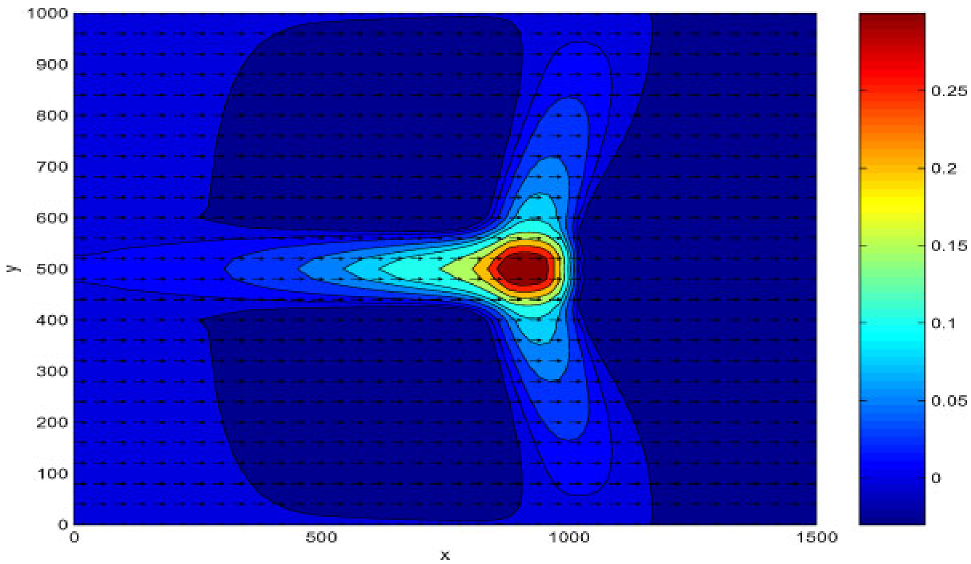

Figure 2. Contour and vector plot of the results using the unsteady approach at $t = 200\,\mathrm{h}$ (B).

When Hudson and Sweby [1] compared the approaches in 1D, the steady approach was more accurate than the unsteady approach when the bed was interacting slowly with the water flow, $A \leqslant 0.01$ and for a small Froude number. However, in 2D the steady approach now seems to be producing inaccurate results due to the kinks present and the difference in position and shape of the pulse.

5. CONCLUSION

In this paper, we have illustrated that in two dimensions, the results of the steady approach are less accurate than in one dimension. Moreover, the steady approach produced worse results than the unsteady approach even for a test case that it was designed for. Thus, in 2D the robustness of the unsteady approach is even more apparent due to the steady approach producing inaccurate results even for test cases where the assumptions made are valid. Unfortunately, for the present unsteady approach the time step size can be a couple of orders of magnitude smaller (depending on the Froude number) than the steady approach, hence leading to associated longer run times. However, we can significantly reduce the run times of the unsteady approach by the adoption of an implicit version of the scheme.

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